

# GROUP ANALYSIS AND RENORMGROUP SYMMETRIES

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An original regular approach to constructing special type symmetries for boundary value problems, namely renormgroup symmetries, is presented. Different methods of calculating these symmetries, based on modern group analysis are described. Application of the approach to boundary value problems is demonstrated with the help of a simple mathematical model.

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## I. INTRODUCTION

The paper is devoted to the problem of constructing a special class of symmetries for boundary value problems (BVPs) in mathematical physics, namely **renormalization group symmetries** (hereafter referred to as RG-symmetries).

Symmetries of this type appeared about forty years ago in the context of the renormalization group (RG) concept. This concept originally arose [1–4] in the "depth" of quantum field theory (QFT) and was connected with a complicated procedure of renormalization, that is 'removing of ultra-violet infinities'. In QFT renormalization group was based upon finite Dyson transformations and appeared as a continuous group in a usual mathematical sense. It was successfully used for improving approximate perturbation solution to restore a correct structure of a solution singularity.

In the seventies, it was found that the RG concept was fruitful in some other fields of microscopic physics: phase transitions in large statistical systems, polymers, turbulence, and so on. However, in some cases, following Wilson's approach [5] to spin lattice, the original *exact symmetry* underlying the renormalization group notion in QFT was changed to an approximate one with the corresponding transformations forming a semi-group (not a group as in the QFT case). Here, in this paper, by RG-symmetry we mean the original exact property of a solution – as it was formulated in Refs. [3,4] (see also [6]) by N. Bogoliubov and one of the present authors. Thus, *by RG-symmetry we mean a symmetry that characterizes a solution of a BVP and corresponds to transformations involving both "dynamical" (i.e., equation) variables and parameters entering into a solution via equations and boundary conditions.*

For a simple illustration we consider some BVP that produces a family of solutions. The simplest variant of RG transformation is given by a simultaneous one-parameter point transformation

$$T_a : \{x \rightarrow x' = x/a, g \rightarrow g' = G(a, g)\} , \quad G(1, g) = g \quad (1)$$

of a dimensionless "coordinate"  $x$  and a one-argument "characteristic" (e.g., initial value)  $g$  of each solution, the quantity of a direct physical interest. The transformation function  $G(x, g)$ , which depends upon two arguments [7], should satisfy the functional equation

$$G(x, g) = G(x/a, G(a, g)), \quad (2)$$

that guarantees the group property  $T_a \cdot T_b = T_{ab}$  fulfillment.

The functional equation (2) and transformation (1) arise, for example, in the massless QFT with one coupling. In that case  $x = Q^2/\mu^2$  is the ratio of a 4-momentum  $Q$  squared to a "reference momentum"  $\mu$  squared,  $g$  is the coupling constant and  $G$  is the so-called *effective coupling*.

Later on the (1)-(2)-type symmetry underlying the renormgroup invariance was also found in a number of problems of macroscopic physics like, e.g., mechanics, transfer theory, hydrodynamics and a close relation of RG-symmetry to the notion of self-similarity was established [9,10].

The infinitesimal form of transformation (1) can be written down as a differential equation

$$RG = 0, \quad \text{with } R = x\partial_x - \beta(g)\partial_g, \quad \beta(g) = \frac{\partial G(a;g)}{\partial a} \Big|_{a=1} \quad (3)$$

where  $R$  is the infinitesimal operator of RG-symmetry (or, simply, *RG-operator*) with the coordinate  $\beta(g)$  defined by the derivative of the function  $G$ .

Therefore, instead of relations (1) and (2), RG transformation can also be introduced by means of an RG-operator. And vice versa, being given an RG-operator one can reconstruct the functional equation for the solution  $G$  with the help of the characteristic equation for (3). Moreover, for a given  $\beta$ -function or, in other words, for the given RG-symmetry, one can get an explicit expression for the invariant of the group transformation  $G(x, g)$  by solving the corresponding Lie equations [11]

$$-\frac{dx'}{x'} = \frac{dg'}{\beta(g')} = \frac{da}{a}, \quad (4)$$

with the initial conditions  $x'|_{a=1} = x$ ,  $g'|_{a=1} = g$ .

Along with (3) a different form of the *invariance condition* for the function  $G(x, g)$  is often employed

$$\frac{d}{da} G(x/a, G(a, g)) \Big|_{a=1} = 0. \quad (5)$$

Equation (3), reflecting the invariance of  $G$  under the RG transformation can be treated as a vanishing condition for the coordinate  $\mathfrak{x}$  of the RG operator (3) in the canonical form [12]

$$\bar{R} = \mathfrak{x}\partial_G, \quad \mathfrak{x} \equiv xG_x - \beta(g)G_g = 0, \quad (6)$$

identically valid on a particular BVP solution  $G = G(x, g)$ .

At the same time, the relation

$$RS(x, g) \equiv (x\partial_x - \beta(g)\partial_g) S(x, g) = \gamma(g)S(x, g) \quad (7)$$

corresponds to the function  $S(x, g)$  that is a *covariant* [13] of the RG transformation. In QFT case, this relates, e.g., to a propagator amplitude (see, Refs. [4] and [6]). Here,  $\gamma(g)$  is known as the *anomalous dimension* of  $S(x, g)$ .

Generally, the differential equation akin to (3)

$$xf_x - \beta(g)f_g = 0. \quad (8)$$

states an invariance of a function  $f$  under the RG transformation (1). Its solution  $f(x, g) = F(G(x, g))$  precisely corresponds to the same property emphasized by (2).

In a particular case, when the function  $G$  is linear in the last argument,  $G \sim kg$ , equation (2) defines a solution that has a power  $x$  dependence, i.e.,  $G(x, g) = gx^k$  with  $k$  being an arbitrary number. Then, equation (2), takes a form of power scaling (or *power self-similarity*) transformation

$$x' = x/a, \quad g' = ga^k,$$

that is well-known in mathematical physics and widely used in the problems of hydrodynamics of liquids and gases.

Therefore, transformation (2) can be considered [10] as a functional generalization  $gx^k \rightarrow G(x, g)$  of a usual (i.e., power) self-similarity transformation. One can refer to it as to *functional self-similarity transformation*: this term was first introduced in [9] as a synonym of the RG transformation as defined above.

It is widely known that in QFT, as well as in other mentioned fields of theoretical physics, the RG method allows one to improve the perturbation theory results and to simplify the analysis of a singular behavior of a solution which becomes scale-invariant in the vicinity of a singularity. The latter reminds a situation, which is typical of asymptotic analysis of solutions of differential equations (DEs): long-time asymptotics demonstrate self-similar regimes [14].

Hence, it looks natural to use the RG methods to study strong nonlinear regimes and to investigate asymptotic behavior of physical systems described by DEs. We have no possibility to discuss this in detail here and would limit ourselves to mentioning some successful attempts of using the RG ideas in mathematical physics.

To our knowledge, the very first results in this field were obtained about a decade ago by two of the co-authors [15] of this paper by applying RG ideas to a problem of generating of higher harmonics in plasma. This problem, after some simplification, was reduced to a couple of partial DEs with the boundary parameter – solution “characteristic” – explicitly included. It was proved that these DEs admit an exact symmetry group similar to that, defined by Eq.(3). The group obtained was then utilized to construct the desired nonlinear solution of the BVP. This approach has further been developed (see [16] and references therein) and we shall describe it in some detail in the next Section.

The methods of QFT RG were exploited by Goldenfeld, Martin and Oono with co-authors (Urbana group) ( see, e.g., Refs. [17] and [18]) to find asymptotics of the solutions of parabolic-type nonlinear differential equations, that describe a variety of physical phenomena, such as groundwater flow under gravity, shock waves dynamics, radiative heat transfer and so on. As an auxiliary tool, they used the concept of *intermediate asymptotics* first introduced by Barenblatt and Zeldovich [19] – see also the review monographs [14] and [20]. In this way, the Urbana team was able to determine values of exponents in the ratios of invariants forming arguments of self-similar solutions.

Later on, with the goal to a global asymptotic analysis they developed and illustrated, by numerous examples, the “perturbative renormalization group theory” (see [21] and references therein) that exploited the form of the invariance condition akin to that (5) used in QFT. The geometrical formulation of the perturbative RG theory for global analysis was presented by Kunihiro [22] on the basis of a classical theory of envelopes.

On the other hand, Bricmont and Kupiainen [23] – [25] attracted RG ideas for nonlinear DEs analyzing in a bit different manner. They used an iterative set of rescalings borrowed from the Wilson version of renormalization group, that is semi-group. On basis of that RG-mapping procedure they succeeded in proving the global existence and detailed long time asymptotics for classes of nonlinear parabolic equations.

Generally, the procedure of revealing RG transformations, or some group features, similar to RG regularities, in any partial case (QFT, spin lattice, polymers, turbulence and so on) up to now is not a regular one. In practice, it needs some imagination and atypical manipulation (see discussion in [8,26,17]) “invented” for every particular case. For example, the above described RG methods applied to asymptotic analysis of differential equations were based on the a priori assumption of the existence of some scaling transformations or on the invariance condition of an approximate solution. By this reason, the possibility to find a regular approach to constructing RG-symmetries is of principal interest. In this paper we give an account of our efforts for creating a possible scheme of this kind in application to physical systems that are described by DEs. The leading idea in this case is based on the fact that symmetries of such systems can be found in a regular manner by using the well-developed methods of modern group analysis.

The paper is organized as follows: in Section II we describe the general scheme of constructing RG-symmetry for a BVP. It appears, that the implementation of this scheme strongly depends on the mathematical model used and on the form of boundary conditions. As a result, different approaches to finding RG-symmetries are possible, and these are illustrated by examples in the following five sections. In Section III RG-symmetries are calculated using the classical Lie symmetries. In Section IV RG-symmetries are obtained on the basis of Lie-Bäcklund symmetries. In the next two sections RG-symmetries are found when boundary conditions are presented either in the form of a differential constraint (Section V), or in the form of an embedding equation (Section VI). In Section VII one more approach to RG-symmetries constructing is presented which is based on an approximate group symmetry. In conclusion, we make a summary of the approach and discuss some further applications.

## II. APPROACH TO CONSTRUCTING RG-SYMMETRIES

First of all, we emphasize that the desired regular approach to constructing RG-symmetries turns out to be possible for those mathematical models of physical systems that are based on differential or, in some particular cases, integro-differential equations. The key idea uses the fact [27,28] that such models can be analyzed by algorithms of modern group analysis.

The proposed scheme comprises a sequence of the four steps.

**I.** A specific manifold (differential, integro-differential, etc.) should be primarily constructed. This manifold that will be referred to as *renormgroup manifold* (RG-manifold) generally differs (see below) from the manifold given by the original system of DEs.

**II.** The second step consists in calculating the most general symmetry group  $\mathcal{G}$  admitted by the RG-manifold.

**III.** The restriction of the group  $\mathcal{G}$  on the desired BVP solution (exact or approximate) constitutes the next step. The group of transformations thus obtained (renormgroup) is characterized by a set of infinitesimal operators (RG-operators), each containing the solution of a BVP in its invariant manifold.

**IV.** The last, fourth step implies utilization of RG-operators to find analytical expressions for solutions of the BVP.

Being formulated in a concise form these steps deserve further comments.

*Comment to I.* In the scheme described above the first step, namely constructing the RG-manifold, is of fundamental importance. The form of its realization depends both on a mathematical model and on a form of a boundary condition. Here we show the following different approaches to RG-manifold constructing:

**Ia.** In the first, more simple case the RG-manifold, as usual in classical group analysis, is presented by a system of basic DEs with the only substantial difference: parameters, entering into a solution via the equation and boundary conditions, are included in the list of independent variables.

**Ib.** Another approach to constructing the RG-manifold implies an extension of a space of variables involved in group transformations, for example, by including differential variables of higher order and nonlocal variables. It means that in this case Lie-Bäcklund transformation groups and nonlocal transformation groups should be invoked [12,29].

**Ic.** In the third approach the procedure of construction of RG-manifold is based on the invariant embedding method [30]. Here RG-manifold is given by a system of equations that consists of original DE and/or embedding equations which correspond to the BVP under consideration.

**Id.** The fourth approach to some extent is similar to the previous one. In this event boundary conditions are reformulated in terms of a differential constraint which is then combined with original equations to form the desired RG-manifold.

**Ie.** The last approach utilizes approximate transformation groups. Here, the RG-manifold is given by a system of DEs with small parameters and can be analyzed by perturbation methods [31].

*Comment to II.* Searching the symmetry of RG-manifold is the main problem of the second step. The term "symmetry" as used in the classical group analysis means the property of a system of DEs to admit a Lie group of point transformations in the basic space of all independent and dependent (differential) variables entering these DEs. The Lie calculational algorithm of finding such symmetries is reduced to constructing tangent vector fields with coordinates, that are functions of these basic group variables and can be defined from the solution of an overdetermined system of DEs, named as *determining equations*. In modern group analysis different modification of a classical Lie scheme are in use (see, e.g. [29,32] and references therein). If the problem of finding symmetries for a given system of DEs (RG-manifold) is solved, then the result is presented in the form of Lie algebra of infinitesimal operators (also known as group generators), which correspond to the admitted vector field. In what follows these operators will be denoted by  $X$ .

*Comment to III.* The goal of a group restriction is the construction of a transformation group with a tangent vector field (point, Lie-Bäcklund, etc.) infinitesimal operators of which (hereinafter referred as  $R$ ) contain the desired BVP solution in an invariant manifold. This means, that the coordinate of the canonical operator of RG-symmetry vanish on the BVP solution and on its differential consequences.

Mathematically, the procedure of a group restriction appears as a "combining" of different coordinates of group generators  $X$  admitted by the RG-manifold. The vanishing condition for this combination on a solution of the BVP leads to algebraic equalities that couple different coordinates and give rise to desired RG-symmetries. In a particular case, when RG is constructed from a Lie group admitted by the original system of DEs, it turns out to be a subgroup of this group and a solution of the BVP appears as an invariant solution with respect to the point RG obtained (compare with [11]). In the general case, not only Lie point group, but Lie-Bäcklund groups, approximate groups, nonlocal transformation groups, etc. (see, e.g. [29]), are also employed as basic groups which are then to be restricted on the solution of a BVP.

*Comment to IV.* A technique for constructing group invariant solutions corresponding to a symmetry group when its infinitesimal operators are known has been detailed in various monographs (see, e.g., [11,32,29]). Therefore, the final step is performed in a usual way and needs no specific comments.

Before proceeding any further, we make a short review of results, that were obtained on the basis of the formulated scheme. The first application of RG-approach to a particular problem of laser plasma was announced in [15]. This problem, namely the problem of a nonlinear interaction of a powerful laser radiation with inhomogeneous plasma, has been detailed in subsequent publications [16,33,34]. A mathematical model was given by a system of nonlinear DEs for components of electron velocity, electron density and the electric and magnetic fields. The presence of small parameters (such as weak inhomogeneity of the ion density, low electron thermal pressure and small angles of incidence of a laser beam on plasma surface) in the initial system of equations provided a way to constructing RG-manifold using (**Ie**) approach, based on approximate group methods. The desired RG-symmetry appears as Lie point symmetry that takes account of transformations of a boundary parameter (common to (**Ia**) approach), which is related to the amplitude of the magnetic field at a critical density point. RG-symmetry obtained made it possible to get the exact solution of original equations, that was then used to evaluate the efficiency of harmonics generation in cold and hot plasma (see [16]).

The advantageous use of the RG-approach in solving the above particular problem gave promise that it may work in other cases. This was illustrated in [35] by a series of examples for different BVPs. Various methods of constructing RG-symmetries were described, based on the use of point symmetries, approximate symmetries, embedding equations and transformations of Fourier components. As is shown in [35], different formulations of BVPs give rise to various methods of finding RG-symmetries. Thus, further development of the scheme was concentrated on analyzing these methods.

The first one was concerned with the initial value problem for the modified Burgers equation with parameters of nonlinearity and dissipation included explicitly. This example [36] yielded a detailed illustration of the method of constructing RG-symmetries when a basic RG-manifold is given by an original DE with parameters included in the list of independent variables (**Ia** approach). It was argued that the exact solution can be reconstructed from the perturbative solution with the help of any of the admitted RG-symmetry operators which form an eight-dimensional algebra. Two illustrative examples were given, dealing with perturbation theory in time and in nonlinearity parameter.

To demonstrate the method of constructing Lie-Bäcklund RG-symmetries that uses (**Ib**) approach, the initial value problem for a linear parabolic equation was considered in [27]. It was shown that appending Lie-Bäcklund RG-symmetries to point RG-symmetries extends the algebra of RG-symmetries up to an arbitrary order.

The same mathematical model was also employed within the (**Ic**) approach when the boundary condition is described by a differential constraint [37]. It was found that RG-symmetries obtained can not be reduced to point RG-symmetries which arise from the (**Ia**) case. However, some of them can be reformulated in terms of RG-symmetries previously found in (**Ia**) approach while the others can be constructed from the Lie-Bäcklund symmetries of basic equations in view of the given differential constraint.

An idea of the (**Id**) approach to constructing RG-manifold based on the invariant embedding method was realized in [35] for ordinary DEs. Here, embedding equations can be treated as a specific form of a differential constraint, that takes boundary data into account. The method of finding RG-symmetry using the (**Id**) approach proves to be of particular interest for the first order ordinary DEs when using (**Ia**) approach faces standard problems in calculating point symmetries admitted by RG-manifold. Provided the embedding equation has the form of a first order DE of an evolutionary type there appear no difficulties in group analysis of a joint system of the basic and the embedding equations.

Worthy of mention is an example that demonstrates the utilization of RG-symmetries to constructing solutions of the BVP for a system of two first-order partial DEs that describes the propagation of a laser beam in a nonlinear focusing medium [38]–[43]. It was revealed that RG-symmetries are related to formal symmetries that are constructed in the form of infinite series in medium nonlinearity parameter. For a specific form of boundary data infinite series are truncated with RG-symmetries presented by finite sums. Generally, for arbitrary boundary data this is not the case and in that event a finite sum describes approximate RG-symmetry for small nonlinearity parameter. Based on (**Ia**), (**Ib**) and (**Ie**) approaches both point and Lie-Bäcklund (exact and approximate) RG-symmetries were obtained and then used to find an analytical solution of the problem.

To clarify the idea of constructing RG-symmetries several examples are given below which demonstrate different approaches to the problem. To gain better understanding of these approaches, a simple mathematical model is used [44]. This model corresponds to BVP for a system of two first-order partial DEs that were studied by Chaplygin [45] in gas dynamics

$$\begin{aligned} v_t + vv_x - a\varphi(n)n_x &= 0, \quad n_t + vn_x + nv_x = 0; \\ v(0, x) &= V(x), \quad n(0, x) = N(x), \end{aligned} \tag{9}$$

where  $\varphi(n)$  is an arbitrary function of  $n$  and  $a$  is a nonlinearity parameter. Despite its simplicity, this mathematical model has a wide field of application and was used to describe various physical phenomena (in the so-called quasi-gaseous media [46]). In such a case the physical meaning of variables  $t$ ,  $x$ ,  $v$  and  $n$  may differ from that in gas dynamics. For example, in nonlinear geometrical optics,  $t$  and  $x$  are coordinates, respectively, along and transverse to the direction of laser beam propagation,  $v$  is the derivative of eikonal with respect to  $x$ , and  $n$  is a laser beam intensity. In this case, functions  $V$  and  $N$  characterize the curvature of the wave front and the beam intensity distribution upon the coordinate  $x$  and the entrance of a medium  $t = 0$ .

Along with (9) another form of basic equations will be used

$$a\tau_v - (n/\varphi(n))\chi_n = 0, \quad \chi_v + \tau_n = 0. \tag{10}$$

These linear equations for new variables  $\tau = nt$  and  $\chi = x - vt$  results from (9) under hodograph transformations.

### III. RG AS LIE POINT SUBGROUP

This section presents an illustration of the method of constructing RG-symmetries when a basic RG-manifold is given by the original DEs with parameters included in the list of independent variables. Boundary conditions are taken into account while restricting the group admitted by RG-manifolds up to the desired RG on the exact or approximate solution of a BVP which thus appears as an invariant solution with respect to any of RG operators obtained.

**3.1.** First we shall consider a particular case of equations (9) when the nonlinearity parameter  $a$  is equal to zero: in application to optical equations discussed above this means that nonlinear effects are neglected. Then by introducing a new variable  $v = \varepsilon u$ , the system of equations (9) is rewritten in the following form

$$u_t + \varepsilon u u_x = 0, \quad n_t + \varepsilon u n_x + \varepsilon n u_x = 0; \quad (11)$$

$$u(0, x) = U(x), \quad n(0, x) = N(x). \quad (12)$$

The continuous point Lie group admitted by the differential manifold (11) (RG-manifold) is given by the infinitesimal operator (a general element of Lie algebra) with six independent terms

$$X = \xi^1 \partial_t + \xi^2 \partial_x + \xi^3 \partial_\varepsilon + \eta^1 \partial_u + \eta^2 \partial_n \equiv \sum_{i=1}^6 X_i, \quad (13)$$

$$X_1 = (1/\varepsilon) \Delta J^1 \partial_t + (J^1 + u \Delta J^1) \partial_x - n J_\chi^1 \partial_n, \quad \Delta J^k \equiv (\varepsilon t J_\chi^k - J_u^k),$$

$$X_2 = (1/n) J^2 (\partial_t + \varepsilon u \partial_x), \quad X_3 = n J^3 \partial_n, \quad X_4 = J^4 (-t \partial_t + n \partial_n + \varepsilon \partial_\varepsilon),$$

$$X_5 = \Delta J^5 D + J^5 (\varepsilon t \partial_x + \partial_u), \quad X_6 = -(1/n) J^6 D, \quad D \equiv (t \partial_t + \varepsilon u t \partial_x - n \partial_n).$$

Coordinates  $\xi$  and  $\eta$  of this infinite-dimensional group operator depend upon five functions  $J^i(\chi, u, \varepsilon)$ ,  $i = 1, 2, 3, 5, 6$  which appear as arbitrary functions of their arguments  $\chi = x - vt$ ,  $u$  and  $\varepsilon$ . The sixth one,  $J^4$ , that enters the operator which describes group transformation of parameter  $\varepsilon$ , is an arbitrary function of this parameter only. The restriction of the group admitted by RG-manifold (11) on the solution of the BVP  $u = \bar{u}(t, x, \varepsilon)$ ,  $n = \bar{n}(t, x, \varepsilon)$  leads to zero equalities for two coordinates of the operator (13) in the canonical form – the conditions of functional self-similarity:

$$\eta^1 + \xi^1 \varepsilon \bar{u} \bar{u}_x - \xi^2 \bar{u}_x - \xi^3 \bar{u}_\varepsilon = 0, \quad \eta^2 + \xi^1 \varepsilon (\bar{n} \bar{u})_x - \xi^2 \bar{n}_x - \xi^3 \bar{n}_\varepsilon = 0. \quad (14)$$

These equalities should be valid for any values of  $t$ , and certainly for  $t = 0$ , when dependencies  $\bar{u}$  and  $\bar{n}$  upon  $x$  are given by boundary conditions (12). This yields two linear relations between  $J^i$  and  $J_\chi^1$ :

$$J^5 = U_x J^1, \quad J^6 = N_x J^1 + N J_\chi^1 - N(U_x)_u J^1 - \varepsilon U_x J^2 - N J^3 - N J^4. \quad (15)$$

Here, and in what follows functions  $U$  and  $N$  and their derivatives with respect to  $x$  should be expressed either in terms of  $u$  or in terms of  $\chi$ . Substituting (15) in (13) gives the desired RG-symmetries with the RG-operator

$$R = \sum_{i=1}^4 R_i, \quad (16)$$

$$\begin{aligned} R_1 = X_1 + & \left[ \left( \varepsilon t (U_x)_\chi - \left( 1 - \frac{N}{n} \right) (U_x)_u - \frac{N_x}{n} \right) J^1 \right. \\ & \left. + \left( \varepsilon t U_x - \frac{N}{n} \right) J_\chi^1 - U_x J_u^1 \right] D + U_x J^1 (\varepsilon t \partial_x + \partial_u), \\ R_2 = X_2 + & \frac{\varepsilon U_x}{n} J^2 D, \quad R_k = X_k + \frac{N}{n} J^k D, \quad k = 3, 4. \end{aligned}$$

We see that RG-symmetries for (11), (12) are presented as a combination of symmetries of infinite-dimensional algebra with the infinitesimal operator (13). Any of the four operators  $R_k$  (and their linear combinations with coefficients

that are arbitrary functions of  $\varepsilon$ ) contains the BVP solution  $u = \bar{u}(t, x, \varepsilon)$  and  $n = \bar{n}(t, x, \varepsilon)$  in the invariant manifold and enables to obtain group transformation of both group variables and different functionals of the solution (for a method of calculating transformation of a functional see Ref. [47]).

Generally, the renormalization group using is capable of improving a perturbation theory solution. As an example, consider the perturbative solution of (11), (12) for small value of  $\varepsilon t \ll 1$

$$u = U(x) - (\varepsilon t)UU_x + O(\varepsilon^2 t^2), \quad n = N(x) - (\varepsilon t)(UN_x + NU_x) + O(\varepsilon^2 t^2). \quad (17)$$

This approximate solution in the limit  $(\varepsilon t) \rightarrow 0$  is invariant with respect to RG transformation defined by the operator  $R_2$  with arbitrary  $\varepsilon J^2 \neq 0$ . Assuming  $J^2 = 1/\varepsilon$ , we obtain the explicit expression for RG-operator

$$R = \frac{1}{n} [(1 + \varepsilon t U_x)(\partial_t + \varepsilon u \partial_x) - \varepsilon U_x n \partial_n], \quad (18)$$

and invariance conditions written in the form of two first order DEs:

$$u_t + \varepsilon u u_x = 0, \quad (1 + \varepsilon t U_x)(n_t + \varepsilon n_x) + \varepsilon n U_x = 0. \quad (19)$$

Solving Lie equations which correspond to RG-operator (18) (and coincide with characteristics equations for (19)) enables to reconstruct the desired exact solution of (11), (12) from the perturbative solution (17)

$$u = U(x - \varepsilon u t), \quad n = \frac{1}{1 + \varepsilon t U_x} N(x - \varepsilon u t), \quad (20)$$

where  $U_x$  should be expressed in terms of  $u$ . For example, in particular case of  $N(x) = N_0 \exp(-x^2)$ ,  $U(x) = -x$  and  $\varepsilon = 1/T$  the latter formulas describe the focusing of gaussian laser beam in geometrical optics

$$n = \frac{T}{T-t} N_0 \exp\left(-x^2 \left(\frac{T}{t-T}\right)^2\right), \quad u = x \frac{T}{t-T}, \quad t \leq T. \quad (21)$$

**3.2.** Now let us turn to a more general case of  $a \neq 0$ . The Lie point symmetry group, admitted by RG-manifold (10), is characterized by a canonical infinitesimal operator [12] with six independent terms  $X_i$ ,  $i = 1, \dots, 5$  and  $X_\infty$

$$X = X_\infty + \sum_{i=1}^5 c_i X_i \equiv \left(\bar{f} + \sum_{i=1}^5 c_i f_i\right) \partial_\tau + \left(\bar{g} + \sum_{i=1}^5 c_i g_i\right) \partial_\chi, \quad (22)$$

where coordinates  $f_i$  and  $g_i$  are linear combinations of  $\tau$  and  $\chi$  and their first derivatives  $\tau_1 = (\partial\tau/\partial n)$  and  $\chi_1 = (\partial\chi/\partial n)$  with coefficients depending only on  $v$  and  $n$  [40,41]. For a particular case  $\varphi = 1$  they are

$$\begin{aligned} f_1 &= \tau, \quad g_1 = \chi; \quad f_2 = -(1/a)n\chi_1, \quad g_2 = \tau_1; \\ f_3 &= -\tau/2 + n\tau_1 + (1/2a)nv\chi_1, \quad g_3 = -(v/2)\tau_1 + n\chi_1; \\ f_4 &= -(1/2)n\chi + v n\tau_1 + [(1/4a)v^2 - n] n\chi_1, \\ g_4 &= (a/2)\tau + (1/2)v\chi + v n\chi_1 + [an - (1/4)v^2] \tau_1; \\ f_5 &= (n\tau_1 - \tau) - a\tau_a, \quad g_5 = n\chi_1 - a\chi_a. \end{aligned} \quad (23)$$

"Evident" symmetries  $f_1$ ,  $g_1$  and  $f_2$ ,  $g_2$  describe dilations of  $\tau$  and  $\chi$  and translations along  $v$ -axis respectively for an arbitrary nonlinearity  $\varphi(n)$ . Two more symmetries  $f_3$ ,  $g_3$  and  $f_4$ ,  $g_4$  appear due to a special form of the function  $\varphi = 1$  under consideration. The symmetry  $f_5$ ,  $g_5$  involves the parameter  $a$  transformation along with transformations of dynamic variables.

The operator  $X_\infty$  with coordinates  $\bar{f} = \xi^1(v, n)$ ,  $\bar{g} = \xi^2(v, n)$  that are arbitrary solutions of partial DEs

$$\xi_v^1 - (n/a)\xi_n^2 = 0, \quad \xi_v^2 + \xi_n^1 = 0, \quad (24)$$

results from the linearity of basic Eqs.(10); it is an ideal of an infinite-dimensional Lie algebra  $L_\infty$  formed by operators  $X_1, \dots, X_5$  and  $X_\infty$ .

The restriction of the group (23) on the BVP solution means that coordinates  $f$  and  $g$  of the canonical operator (22) turns to zero on this solution, that is

$$\bar{f} = -\sum_{i=1}^5 c_i f_i, \quad \bar{g} = -\sum_{i=1}^5 c_i g_i. \quad (25)$$

These relations express functions  $\bar{f}$  and  $\bar{g}$  in terms of  $f_i$ ,  $g_i$ ,  $i = 1, \dots, 5$  taken on a solution  $\tau = \bar{\tau}(v, n)$ ,  $\chi = \bar{\chi}(v, n)$  of a BVP (exact or approximate). Substitution of (25) in (22) gives five RG-operators

$$R = \sum_{i=1}^5 c_i(a) R_i, \quad (26)$$

each being determined by corresponding coordinates  $f_i$ ,  $g_i$  and by a pair of functions  $A^i$ ,  $B^i$

$$\begin{aligned} R_1 &= (\tau - A^1) \partial_\tau + (\chi - B^1) \partial_\chi, \quad R_2 = -A^2 \partial_\tau - B^2 \partial_\chi + \partial_v, \\ R_3 &= (-\tau/2 - A^3) \partial_\tau - B^3 \partial_\chi - (v/2) \partial_v - n \partial_n, \\ R_4 &= (-(n/2) \chi - A^4) \partial_\tau + ((a/2) \tau + (v/2) \chi - B^4) \partial_\chi \\ &\quad + (-(1/4) v^2 + a n) \partial_v + v n \partial_n, \\ R_5 &= (-\tau - A^5) \partial_\tau - B^5 \partial_\chi - n \partial_n + a \partial_a \end{aligned} \quad (27)$$

Here, ten functions  $A^i, B^i$  are defined by expressions (23) for  $f^i$  and  $g^i$  where one should replace  $\tau$ ,  $\chi$  by  $\bar{\tau}(n, v)$ ,  $\bar{\chi}(n, v)$ . Explicit formulas for RG-operators depend upon the specific solution of the BVP. For example, for the particular solution of the BVP (9) with  $V = 0$  and  $N(x) = \cosh^{-2}(x)$ , described by [48]

$$\begin{aligned} \tau &= \frac{(v/2)^{1/2}}{a^{3/4}} \left( \sqrt{\kappa^2 + 1} - \kappa \right)^{1/2}, \quad \kappa = \frac{\sqrt{a}}{v} \left( 1 - n - \frac{v^2}{4a} \right), \\ \chi &= -\frac{1}{2} \ln \frac{(v/2\sqrt{a})^{1/2} + (\sqrt{\kappa^2 + 1} - \kappa)^{1/2}}{-(v/2\sqrt{a})^{1/2} + (\sqrt{\kappa^2 + 1} - \kappa)^{1/2}}, \end{aligned} \quad (28)$$

functions  $A^5$ ,  $B^5$  in (27) are expressed as follows [40]:

$$\begin{aligned} A^5 &= -\frac{(v/2)^{1/2}}{4a^{3/4}\sqrt{1+\kappa^2}} \left( \sqrt{\kappa^2 + 1} - \kappa \right)^{1/2} \left( \kappa + \sqrt{1+\kappa^2} - 2\frac{\sqrt{a}}{v} \right), \\ B^5 &= \frac{(v/2)^{1/2}}{4a^{1/4}\sqrt{1+\kappa^2}} \frac{(\sqrt{\kappa^2 + 1} - \kappa)^{1/2}}{(\sqrt{\kappa^2 + 1} - \kappa - (v/2\sqrt{a}))} \left( \sqrt{1+\kappa^2} - 3\kappa + 2\frac{\sqrt{a}}{v} - \frac{v}{\sqrt{a}} \right). \end{aligned}$$

It should be noticed, that the solution of the presented above BVP is unique, but a number of RG-operators that give rise to this solution is different from one (in the first example case we have four RG-operators with arbitrary functions of  $(n, \chi)$ , and in the second example five RG-operators with arbitrary functions of  $a$ ). In the next section we will show that the number of RG operators may be enlarged to an arbitrary value, provided not only point but Lie-Bäcklund groups are taken into account.

#### IV. RG AS LIE-BÄCKLUND SUBGROUP

The method of constructing RG-symmetries from Lie point symmetries admitted by the original DE is naturally generalized to include Lie-Bäcklund (L-B) symmetries. The extension of the space of differential variables increases the amount of BVPs that allow restriction of a group on their solution. A complete set of RG-symmetries is obtained by appending L-B RG-symmetries to point RG-symmetries. In this section we present an example of constructing

L-B RG-symmetries of the second order for the BVP (9). As in the previous section we use a transformed form of the basic equations (10).

L-B symmetries admitted by the RG-manifold (10) are characterized by the same canonical infinitesimal operator (22) where additional terms proportional to higher-order derivatives of  $\tau$  and  $\chi$  should be added in coordinates  $f$  and  $g$ . Similarly to first-order symmetries, these terms are linear combinations of  $\tau$  and  $\chi$  and their derivatives  $\tau_i = (\partial^i \tau / \partial n^i)$  and  $\chi_i = (\partial^i \chi / \partial n^i)$  with coefficients that depend only on  $v$  and  $n$  [39–41]. For the second-order Lie-Bäcklund symmetries in a particular case  $\varphi(n) = 1$ , we have five additional operators  $X_i$  with  $i = 7, \dots, 11$  (the term with  $i = 6$  corresponds to  $X_\infty$  and is omitted in the sum, i.e.  $c_6 = 0$ )

$$X = X_\infty + \sum_{i=1}^{11} c_i X_i \equiv \left( \bar{f} + \sum_{i=1}^{11} c_i f_i \right) \partial_\tau + \left( \bar{g} + \sum_{i=1}^{11} c_i g_i \right) \partial_\chi. \quad (29)$$

It should be noted that expressions for all coordinates in (29) can be obtained by the action of the following three recursive operators [41]  $L_i$ ,  $i = 1, 2, 3$

$$\begin{aligned} L_1 &= \begin{pmatrix} 0 & -(n/a)D_n \\ D_n & 0 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 2nD_n - 1 & (n/a)vD_n \\ -vD_n & 2nD_n \end{pmatrix}, \\ L_3 &= \begin{pmatrix} 2nvD_n & n(v^2/2a - 2n)D_n - n \\ (-v^2/2 + 2an)D_n + a & 2nvD_n + v \end{pmatrix}, \end{aligned} \quad (30)$$

on the "trivial" operator with  $f = \tau$  and  $g = \chi$  (here,  $D_n$  is the operator of total differentiation with respect to  $n$ ). Below, we present only three of these five second-order L-B operators

$$\begin{aligned} f_7 &= n\tau_2, & g_7 &= \chi_1 + n\chi_2; \\ f_8 &= (1/2a)n[-\chi_1 + v\tau_2 - 2n\chi_2], & g_8 &= (1/2a)v\chi_1 + n\tau_2 + \frac{1}{2a}nv\chi_2; \\ f_9 &= (1/4)\tau - n\tau_1 - (5/4a)vn\chi_1 + (-n + (1/4a)v^2)n\tau_2 - (1/a)vn^2\chi_2, \\ g_9 &= (3/4)v\tau_1 - (2n - (1/4a)v^2)\chi_1 + vn\tau_2 + (-n + (1/4a)v^2)n\chi_2. \end{aligned} \quad (31)$$

The procedure of restriction of the L-B group obtained on the solution of the BVP leads to expressions for  $\bar{f}$  and  $\bar{g}$  akin to (25)

$$\bar{f} = - \sum_{i=1}^{11} c_i f_i, \quad \bar{g} = - \sum_{i=1}^{11} c_i g_i. \quad (32)$$

Substitution of (32) in (29) yields additional terms in the expression (26) for the RG-operator  $R$  that depends on higher-order derivatives of  $\tau$  and  $\chi$

$$R = \sum_{i=1}^{11} c_i(a) R_i \equiv \sum_{i=1}^{11} c_i(a) ((f_i - A^i) \partial_\tau + (g_i - B^i) \partial_\chi). \quad (33)$$

Here functions  $A^i$  and  $B^i$  are given by the corresponding formulas for coordinates  $f_i$  and  $g_i$  to be evaluated on the solution  $\bar{\tau}(n, v)$  and  $\bar{\chi}(n, v)$ . It appears that coordinates of L-B RG-operators are obtained from point RG-operators with the help of the above-mentioned recursive operators, hence, one can obtain L-B RG-operators of an arbitrary high order. Despite an unusual form, we still call them RG-operators since they possess the main property of RG-operators, namely, they contain a solution of the BVP in their invariant manifold.

The procedure of using L-B RG-operators is not as simple as for point RG-operators. Yet we can describe two possible ways.

Firstly, coordinates of canonical L-B RG-operators can be used to construct a set of relations, differential constraints, that are compatible with the original DEs and satisfy specific boundary conditions. The use of such constraints is described in section 6. In the general case, for an arbitrary L-B group of a given order, coordinates of the corresponding

canonical operator can be treated as a set of differential expressions, zero equalities for which impose appropriate restrictions on the basic DEs, consistent either with physical or with symmetry conditions. These equalities can also be treated as embedding equations (see [27]).

Secondly, L-B RG-operators can be used to construct invariant solutions that automatically fit boundary conditions. It should be noticed that in some particular cases, L-B RG-symmetries can be constructed from a L-B group with a finite number of operators. For example, RG-symmetry for (9) with boundary conditions  $V = 0$  and  $N = \cosh^{-2}(x)$  appears as a linear combination of three L-B symmetries

$$R = (f_3 + 2(f_7 + f_9))\partial_\tau + (g_3 + 2(g_7 + g_9))\partial_\chi. \quad (34)$$

The desired solution of the BVP can be found as the invariant solution with respect to RG-operator (34) and is presented by formulas (28).

The recipe of constructing the L-B renormgroup formulated in this section goes far beyond a simple illustrative example for the BVP (9). In a similar way, L-B RG-operators are constructed for different BVPs of mathematical physics that admit L-B symmetries; other examples are presented in [27] for the linear parabolic and modified Burgers equation. It is essential that when parameters entering into the equation and boundary conditions are involved in group transformations, coordinates of canonical L-B RG-operators contain not only first but higher-order derivatives with respect to these parameters. This means that in addition to recursive operators containing operators of total differentiation with respect to  $n$  (for BVP (9)), new recursive operators comprise operators of differentiation with respect to parameters, as well ( $\propto D_\epsilon$  and  $D_a$  in the case of BVPs (11) and (10)).

## V. RG DEVISING BASED ON EMBEDDING EQUATIONS

In this section we present a specific method of constructing RG-symmetries [35,27], which is based on embedding equations [30]. It is of prime interest for physical systems described by ordinary differential equations (ODEs). In the context of the discussed model of quasi-Chaplygin media such equations arise, e.g., when constructing invariant solutions with respect to symmetries obtained. We demonstrate the idea of this method for the very simple BVP

$$u_t = f(t, u, a); \quad t = \tau, \quad u = x. \quad (35)$$

Extension of the original differential manifold by adding, to the original equation, the embedding equation that appears as a linear first-order partial DE

$$u_\tau + f(\tau, x, a)u_x = 0, \quad (36)$$

gives the desired RG-manifold, where  $u$  is now treated as the function of four variables  $\{t, \tau, x, a\}$ . Performing the group analysis for this RG-manifold involves boundary data and parameter  $a$  in group transformations, while the subsequent restriction of the group obtained on any solution of the BVP yields the desired RG-symmetries. We give two examples of such calculations for  $f = au^2$  and  $f = u^2 + au^3$ .

**5.1.** In the event of  $f = au^2$  the RG-manifold (35)-(36) is given by two equations

$$u_t = au^2, \quad u_\tau + ax^2u_x = 0 \quad (37)$$

that admit an infinite-dimensional Lie point algebra with five independent elements

$$X = \sum_{i=1}^5 \alpha_i X_i, \quad (38)$$

$$X_1 = \partial_t + au^2\partial_u, \quad X_2 = \partial_\tau + ax^2\partial_x,$$

$$X_3 = u^2\partial_u, \quad X_4 = x^2\partial_x, \quad X_5 = x^2\tau\partial_x + u^2t\partial_u + \partial_a.$$

Here, functions  $\alpha_1$  and  $\alpha_2$  depend upon five variables  $\{t, \tau, x, a, u\}$ , whereas  $\alpha_i$ ,  $i = 3, 4, 5$  are arbitrary functions of three combinations  $at + (1/u)$ ,  $a\tau + (1/x)$ ,  $a$ .

The procedure of restriction of the group obtained leads to the invariance condition

$$U^2(\alpha_3 + a\alpha_1 + \alpha_5 t) - \alpha_1 U_t - \alpha_2 U_\tau - x^2(\alpha_4 + a\alpha_2 + \alpha_5 \tau)U_x - \alpha_5 U_a = 0 \quad (39)$$

to be fulfilled on an exact or approximate solution  $u = U(t, x, \tau, a)$  of the BVP (35)-(36); for example, one can take the perturbative solution as an expansion in powers of  $a$

$$u = U(t, x, \tau, a) \equiv x + ax^2(t - \tau) + O(a^2), \quad a \ll 1. \quad (40)$$

Substituting (40) into (39) shows that the invariance condition (39) is fulfilled for  $\alpha_3 = \alpha_4 \equiv \alpha$  and arbitrary  $\alpha_1, \alpha_2$  and  $\alpha_5$ . Assuming  $\alpha_1 = \alpha_2 = \alpha = 0$  and  $\alpha_5 = 1$  in (38) yields one of the RG-operators

$$R = x^2 \tau \partial_x + \partial_a + u^2 t \partial_u, \quad (41)$$

which enables us to transform the perturbative solution of (35) for small  $a \ll 1$  to the following exact solution

$$u = \frac{x}{1 - ax(t - \tau)}.$$

This result is found by solving the Lie equations, that correspond to the RG-operator (41).

**5.2.** For another value of the function  $f = u^2 + au^3$ , the RG-operator that is similar to (41) is given as follows

$$R = (x^2(1 + ax)\tau + x) \partial_x + (u^2(1 + au)t + u) \partial_u - a \partial_a. \quad (42)$$

The invariance condition for the solution of the BVP with respect to the RG-operator (42) has the form of the first-order partial DE

$$- (x^2(1 + ax)\tau + x) u_x + au_a + u^2(1 + au)t + u = 0. \quad (43)$$

Solving the characteristic equations for (43) (Lie equations) yields the following exact solution of the BVP (35) with  $f = u^2 + au^3$

$$t - \tau = \frac{1}{x} - \frac{1}{u} + a \ln \left| \frac{x(1 + au)}{u(1 + ax)} \right|.$$

What all renormgroups obtained for the BVPs for the first-order ODE in the above examples have in common is that their operators depend upon arbitrary functions  $\alpha_i$ , which means that RG can be expressed in terms of different RG-operators with various particular expressions for their coordinates. This situation is the same as that one obtains for the BVP in the case of partial DE: different RG-operators yield the same unique specific solution of the given BVP contained in the invariant manifold of RG-operators. The previous procedure of RG constructing for the BVP for the ODE was based on the use of point groups. However, L-B groups can also be employed for constructing RG-symmetries for the first-order ODE, especially, in view of embedding equations (see Refs. in [27]).

The structure of embedding equations depends not only on the form of the original equation, but also on the boundary conditions. This means that for given basic equations we may obtain different embedding equations. For example, if the function  $f$  in the r.h.s. of (35) depends upon  $x$ ,

$$u_t = f(t, x, a, u); \quad t = \tau, \quad u = x \quad (44)$$

we arrive at the embedding equation

$$u_\tau + f(\tau, x, a, x)u_x = f(\tau, x, a, x) + \int_\tau^t dt' f_x(t', x, a, u(t')) \exp \left[ - \int_t^{t'} dt'' f_u(t'', x, a, u(t'')) \right]. \quad (45)$$

Hence, the RG manifold in this case is defined by a system of integro-differential equations (44) and (45) and one should employ the modern group analysis techniques which give a possibility of analyzing such equations, as well [49,50].

## VI. RG AND DIFFERENTIAL CONSTRAINT

In the previous section RG-manifold was obtained by combining an original DE and an embedding equation. More generally instead of an embedding equation, an additional differential constraint can be used that satisfy two conditions: firstly, it must be compatible with the original DE and, secondly, it should explicitly take boundary

conditions into account. This constraint naturally emerges when a coordinate of a canonical operator of the L-B RG admitted by BVP is assumed to be equal to zero. Adding this constraint to original equations we obtain the RG-manifold.

**6.1.** To illustrate, consider first a BVP (9) with  $a = 0$  which we rewrite using hodograph transformations in a simple form (compare with (10))

$$\chi_n = 0, \quad \chi_v + \tau_n = 0. \quad (46)$$

L-B symmetries of this system of DEs are given by a canonical operator

$$X = f\partial_\tau + g\partial_\chi, \quad (47)$$

with coordinates  $f$  and  $g$  depending upon  $v$  and derivatives  $\tau_s + n\chi_{s+1}$ ,  $\chi_s$  of an arbitrary order  $s \geq 0$

$$f = F(v, \chi_s, \tilde{\tau}_s) - n \left[ \partial_v + \sum_{k=0}^{\infty} (\tilde{\tau}_{k+1} \partial_{\tilde{\tau}_s} + \chi_{k+1} \partial_{\chi_k}) \right] G, \quad g = G(v, \chi_s, \tilde{\tau}_s), \quad (48)$$

$$\tilde{\tau}_s = \tau_s + n\chi_{s+1}, \quad \tau_s = (\partial^s \tau / \partial v^s), \quad \chi_s = (\partial^s \chi / \partial v^s).$$

Consider a particular case of a BVP (9) with boundary conditions defined by  $V(x) = -\varepsilon x$  and arbitrary  $N(x)$ . In terms of the variables  $\tau$  and  $\chi$ , these conditions are described, for example, by a pair of differential constraints

$$\chi_{vv} = 0, \quad \tau_{vv} - N_{vv}\chi_v - N_v\chi_{vv} = 0. \quad (49)$$

Here the dependence of  $N$  upon  $x$  is given in terms of  $v$  with the use of the above boundary condition.

It is easily checked by direct substituting into (48) that left-hand sides of these equalities are the corresponding coordinates  $g$  and  $f$  of the second-order L-B symmetry operator (47). Adding differential constraints (49) to the original equation (46), we obtain the desired RG-manifold

$$\chi_n = 0, \quad \chi_v + \tau_n = 0, \quad \chi_{vv} = 0, \quad \tau_{vv} - N_{vv}\chi_v = 0. \quad (50)$$

The latter admits a 17-parameter group of point transformations given by the following operators

$$X = \sum_{i=1}^m c_i X_i, \quad m = 17, \quad (51)$$

$$X_1 = v^2 \partial_v + v(2(n - N) + vN_v) \partial_n + (\chi(N - n) + \tau v) \partial_\tau + v\chi \partial_\chi,$$

$$X_2 = v\chi \partial_v + (\chi(n - N) + v(\chi N_v - \tau)) \partial_n + 2\tau\chi \partial_\tau + \chi^2 \partial_\chi,$$

$$X_3 = -v\partial_v + (N - n - vN_v) \partial_n, \quad X_4 = v\chi \partial_n - \chi^2 \partial_\tau,$$

$$X_5 = v\partial_n, \quad X_6 = (N - n) \partial_n + \chi \partial_\chi, \quad X_7 = (n - N) \partial_n + \tau \partial_\tau,$$

$$X_8 = \partial_\tau, \quad X_9 = v\partial_\tau, \quad X_{10} = (N - n) \partial_\tau + v\partial_\chi,$$

$$X_{11} = \partial_\chi, \quad X_{12} = \chi \partial_\tau, \quad X_{13} = -v^2 \partial_n + v\chi \partial_\tau.$$

$$X_{14} = -\partial_v - N_v \partial_n, \quad X_{15} = \partial_n, \quad X_{16} = \chi \partial_n, \quad X_{17} = -\chi \partial_v + (\tau - \chi N_v) \partial_n.$$

The usual procedure of restriction of the group obtained on a solution of the BVP (46) relates different coefficients in the sum (51) and gives the desired RG operators

$$R = \sum_{i=1}^{13} c_i R_i, \quad (52)$$

$$\begin{aligned}
R_1 &= X_1, & R_2 &= X_2, & R_3 &= X_3 + \varepsilon X_{17}, & R_4 &= X_4, \\
R_5 &= X_5 + \varepsilon X_{16}, & R_6 &= X_6 + \varepsilon X_{17}, & R_7 &= X_7 + \varepsilon X_{16}, \\
R_8 &= X_8 + \varepsilon X_{15}, & R_9 &= X_9 - \varepsilon^2 X_{16}, & R_{10} &= X_{10} - \varepsilon^2 X_{17}, \\
R_{11} &= X_{11} + \varepsilon X_{14}, & R_{12} &= X_{12} + \varepsilon X_{16}, & R_{13} &= X_{13}.
\end{aligned}$$

The exact solution of the BVP  $\chi = -v/\varepsilon$ ,  $\tau = (1/\varepsilon)(n - N)$  is found either by solving Lie equations corresponding to any of these RG-operators, or as the intersection of all invariant manifolds.

**6.2.** Now let us turn to a general case of a BVP (9) with  $a \neq 0$ . We shall consider the problem of constructing RG-symmetries using the RG-manifold given by basic equations in the form (10) and the most simple differential constraint yielded by the linear combination of the second order L-B symmetry (31)  $f_7, g_7$  and trivial infinite-dimensional symmetry  $f_\infty = 0, g_\infty = -1$

$$a\tau_v - n\chi_n = 0, \quad \chi_v + \tau_n = 0, \quad n\tau_{nn} = 0, \quad \chi_n + n\chi_{nn} - 1 = 0. \quad (53)$$

This differential constraint describes, in particular, a linear dependence of  $N$  upon  $x$  and  $V(x) = 0$ . The Lie point group admitted by the RG-manifold (53) is characterized by seven infinitesimal operators (use the formula (51) for  $m = 7$ )

$$\begin{aligned}
X_1 &= -2v\partial_v - 4n\partial_n - 6n(v/a)\partial_\tau + (2\chi - 6n + 3v^2/a)\partial_\chi, \\
X_2 &= v\partial_v + 2n\partial_n + (\tau + 2nv/a)\partial_\tau + (2n - v^2/a)\partial_\chi, \quad X_3 = \partial_v, \\
X_4 &= n\partial_\tau - v\partial_\chi, \quad X_5 = (v/a)\partial_\tau + \ln n\partial_\chi, \quad X_6 = \partial_\tau, \quad X_7 = \partial_\chi.
\end{aligned}$$

The restriction of this group on the solution of the BVP with the above-mentioned boundary conditions leads to the three-parameter RG

$$R_1 = X_1, \quad R_2 = X_2, \quad R_3 = aX_3 + X_4.$$

As in the previous case, the exact solution of the BVP  $\tau = nw$ ,  $\chi = n - aw^2/2$  appears as an intersection of all invariant manifolds that correspond to these RG-operators.

The characteristic feature of the described approach is the formulation of boundary data in the form of a differential constraint and the subsequent search of the group admitted by this constraint and basic equations. It is evident that there exists an infinite number of other differential constraints that adequately describe the same boundary data and the use of which leads to different RG algebras. As an example, we can point to differential constraints that arise from the zero equality of appropriate coordinates of the infinite L-B algebra.

The example of RG-symmetry constructing on the basis of L-B symmetry reveals the practical importance of the latter and, on the other hand, demonstrates point symmetries that are not admitted by the original equation. The procedure of construction of RG-symmetries with the help of a differential constraint was also carried out in [37] for the linear parabolic equation.

## VII. RG AS A SUBGROUP OF AN APPROXIMATE SYMMETRY GROUP

An attractive method of RG constructing is that based on approximate symmetries [31]. This method can be applied to systems described in terms of models based on DEs with small parameters. These small parameters allows us to consider a simple subsystem of the original DEs that usually admits an extended symmetry group inherited by the original DEs. Restricting this approximate group on the solution of the BVP yields the desired RG-symmetries. The merits of the described method is illustrated below for the BVP (9) with a small nonlinearity parameter  $a \ll 1$ . In terms of the variable  $w = v/a$  the following basic system of linear DEs is obtained instead of (10):

$$\tau_w - (n/\varphi(n))\chi_n = 0, \quad \chi_w + a\tau_n = 0. \quad (54)$$

For  $a = 0$ , it admits an infinite L-B symmetry group

$$X = f\partial_\tau + g\partial_\chi, \quad (55)$$

characterized by an arbitrary dependence of the zero-order coordinates  $f = f^0$  and  $g = g^0$  upon  $n, \tau, \chi$  and the derivatives  $\tilde{\tau}_s, \chi_s$  of an arbitrary order

$$f^0 = F^0 + \int dw \{ (n/\varphi) Y g^0 \}, \quad g^0 = G^0. \quad (56)$$

Here and below in (57)

$$Y = \partial_n + \sum_{s=0}^{\infty} (\tau_{s+1} \partial_{\tau_s} + \chi_{s+1} \partial_{\chi_s}),$$

$$\tau_s = \frac{\partial^s \tau}{\partial n^s}, \quad \chi_s = \frac{\partial^s \chi}{\partial n^s}, \quad \tilde{\tau}_s = \tau_s - w \sum_{p=0}^s \binom{s}{p} \frac{\partial^p (n/\varphi)}{\partial n^p} \chi_{s-p+1},$$

$F^i(n, \chi_s, \tilde{\tau}_s)$  and  $G^i(n, \chi_s, \tilde{\tau}_s)$  are arbitrary functions of their arguments, and expressions in curly brackets before integrating over  $w$  should be given in terms of  $\tilde{\tau}_s, \chi_s, n, w$ .

For  $0 < a \ll 1$ , this symmetry is inherited as an approximate one by equations (54) which thus represent an approximate RG-manifold. For example, for  $\varphi(n) = 1$  the following result is obtained:

$$f^i = F^i + \int dw \left\{ Z f^{i-1} + \frac{n}{\varphi} Y g^i \right\}, \quad g^i = G^i + \int dw \{ Z g^{i-1} - Y f^{i-1} \}, \quad (57)$$

$$Z = \sum_{s=0}^{\infty} \tau_{s+1} \partial_{\chi_s}, \quad \tilde{\tau}_s = \tau_s - w(n\chi_{s+1} + s\chi_s), \quad i \geq 1.$$

One can see, that the symmetry of equations (54) for  $a = 0$  is inherited by the symmetry of these equations for  $a \neq 0$  up to an arbitrary order of this parameter. It should be noticed that both zero-order and higher-order approximate symmetries may appear as Lie point symmetries or L-B symmetries, and this parameter may be involved in group transformations, as well.

The restriction of the approximate group obtained on a particular solution of the BVP defines the specific form of the zero-order symmetries. It means that while constructing RG-symmetries for the BVP (54) in view of the boundary data from (9), coordinates  $f^0, g^0$  and "integration constants"  $F^i, G^i, i \geq 1$  are not arbitrary functions, but should be chosen so that relations  $f = 0, g = 0$  satisfy desired boundary conditions  $\tau_s = 0, \chi = H(n)$  at  $w = 0$ . Provided that the functions  $F^i$  and  $G^i, i \geq 1$  are also equal to zero in this case, boundary conditions are correlated with the form of functions  $f^0$  and  $g^0$ . In general, invariance conditions  $f = 0$  and  $g = 0$  appear as differential constraints (or algebraic relations) to be fitted by boundary data.

Of special interest are such zero-order functions  $f^0$  and  $g^0$  for which infinite series (57) are truncated for some finite value of  $i = i_{max}$ , and we arrive at finite sums. In this case, instead of an approximate group with respect to a small parameter  $a$  we obtain the exact symmetry group (compare with [31, §11]). A simple example of this is given by the RG-operator (34). It is easily checked that in terms of  $n$  and  $w$ , the combinations of coordinates  $f_3 + 2(f_7 + f_9)$  and  $g_3 + 2(g_7 + g_9)$  are expressed as binomial in  $a$ , i.e. expressions for  $f$  and  $g$  are represented as zero-order and first-order terms  $f = f^0 + a f^1$  and  $g = g^0 + a g^1$ , where  $f^0, g^0$  and  $f^1, g^1$  according to (56) and (57) are defined by the formulas

$$f^0 = 2n(1-n)\tau_2 - n\tau_1 - 2nw(\chi_1 + n\chi_2),$$

$$g^0 = 2n(1-n)\chi_2 + (2-3n)\chi_1, \quad (58)$$

$$f^1 = \frac{1}{2}nw^2\tau_2, \quad g^1 = 2nw\tau_2 + w\tau_1 + \frac{1}{2}(nw^2\chi_2 + w^2\chi_1).$$

From here, in view of (57), it follows that higher-order corrections vanish, and we obtain an exact second-order L-B symmetry of DEs (54) at  $\varphi = 1$  for arbitrary  $a \neq 0$ ; this symmetry gives rise to the exact solution [38,39] satisfying the boundary condition  $N = \cosh^{-2}(x)$  defined by the zero order term  $g^0$ .

The arbitrariness in functions  $f^0, g^0$  enables us to construct RG-symmetries for any boundary conditions. As an illustration, we present RG-symmetries for the BVP with

$$H(n) = (\ln(1/n))^{1/2}, \quad (59)$$

describing space evolution (self-focusing) of the gaussian beam with the originally plane phase front at  $\tau = 0$ . To satisfy the initial distribution (59), one can choose the following functions  $f^0 = 1 + 2n\chi\chi_1$  and  $g^0 = 0$ . For this value of  $f^0$  the inherited point group of the BVP is constructed with the help of formulas (57) and is given by the operator

$$R = -2\chi\partial_w + 2a\tau\partial_n + \left(1 + \frac{a\tau^2}{n}\right)\partial_\tau. \quad (60)$$

The invariance condition for the solution of the BVP with respect to RG with this operator is presented in the form of two partial DEs

$$\chi\chi_w - a\tau\chi_n = 0, \quad 2\chi\tau_w - 2a\tau\tau_n + 1 + (a\tau^2/n) = 0,$$

the solution of which yields the desired approximate analytical solution of the problem

$$x^2 = (ant^2 - \ln n) \left[1 - 2Q(\sqrt{ant^2})\right]^2, \quad v = -2\frac{x}{t} \frac{Q(\sqrt{ant^2})}{\left[1 - 2Q(\sqrt{ant^2})\right]}. \quad (61)$$

Here the function  $Q(z)$  is expressed as follows

$$Q(z) = ze^{-z^2/2} \int_0^z dt e^{t^2/2}.$$

The first-order approximate symmetry obtained can be used to calculate a higher-order approximation of the RG-operator (60) and, thus, to improve the analytical solution (61). One can also obtain new-type RG-operators just by substituting the approximate solution (61) into formulas (27).

## VIII. CONCLUSION

This paper presents a new approach to constructing RG-symmetries based on the mathematical apparatus of classical and modern group analysis. It differs from the traditionally used methods of constructing RGs in theoretical physics and is formulated as a sequence of the following steps:

I) *constructing the RG-manifold*, that takes into account both basic equations and the corresponding boundary conditions;

II) *calculating the symmetry group*, admitted by RG-manifold;

III) *restricting the group obtained on the solution of the BVP*;

IV) *utilizing of RG-operators to find analytical expressions for solutions*.

As it was shown there exists a set of different algorithms for finding RG-symmetries. The choice of a particular one for a given physical problem depends on a mathematical model used for the problem description.

It should be noted, that different methods of constructing RG-symmetries described above do not exhaust the suggested approach (see, e.g., [42,43]). Procedure of constructing RG-symmetries may combine different algorithms; for example, of interest is a simultaneous use of the method based on approximate symmetries and the invariant embedding method, and so on.

Our approach reveals a close relation of *functional self-similarity* property (i.e., "classical" RG-symmetry as an exact property of a solution) to an invariance condition of a BVP solution with respect to RG-operator. Mathematically, the latter is formulated as the vanishing condition for the coordinate of a canonical RG-operator on a solution of BVP.

One can readily see that RG-operators may appear in the form, that is different from QFT case [6], e.g., operators of Lie-Bäcklund RG-symmetries. However, in some cases "our" RG-operators can look like that ones in QFT renormalization group. For example, linear combination of operators  $\alpha_1 X_1$  and  $\alpha_3(X_3 + X_4)$  for the BVP (35) with  $\alpha_1 = 1$ ,  $\alpha_3 = -a$  gives

$$R = \partial_t - ax^2\partial_x, \quad (62)$$

which is formally equivalent (with appropriate change of variables  $t = \ln x$ ,  $x = g$  and  $\beta(g) = ag^2$ ) to the differential operator for one-coupling massless QFT model in one-loop approximation.

Up to now this approach is feasible for systems that can be described by DEs and is based on the formalism of modern group analysis.

It seems also possible to extend our approach on physical systems that are not described just by differential equations. A chance of such extension is based on recent advances in group analysis of systems of integro-differential equations [49,50] that allow transformations of both dynamical variables and functionals of a solution to be formulated [47]. More intriguing is the issue of a possibility of constructing a regular approach for more complicated systems, in particular to that ones having an infinite number of degrees of freedom. The formers can be represented in a compact form by functional integrals (or *path integrals*).

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